Quiz 3: October 9

Name: Solutions

SID: _____

Problem 1. Let $\mathcal{P}_2(x)$ be the real vector spaces of degree ≤ 2 polynomials. Let $T : \mathcal{P}_2(x) \to \mathbb{R}$ be the linear map given by the formula

$$T(p(x)) = 2p'(0)$$

Show that $\dim(\mathcal{P}_2(x)/\operatorname{null} T) = 1$.

1) $\dim(\text{null}(T)) = \dim(P_2) - \dim(\text{Im}(T)) = 2$] 2 pts

2.) $\dim(V/U) = \dim(V) - \dim(U)$] 2 pts.

$\Rightarrow \dim(P_2/\text{Null}(T)) = 3 - 2 = 1.$

Problem 2. Let $\varphi, \psi : \mathbb{F}^2 \to \mathbb{F}$ be two linear functionals. Assume that

$$\operatorname{null}\varphi \cap \operatorname{null}\psi = \{0\}.$$

Show that φ, ψ form a basis for $(\mathbb{F}^2)'$.

Solution 1: Since $\dim((\mathbb{F}^2)') = 2$, it suffices to show that φ and ψ are linearly independent. Suppose they are not. Then, either $\varphi = \alpha\psi$ or $\psi = \beta\varphi$.
WLOG, let $\varphi = \alpha\psi$. Then, $\operatorname{null}(\varphi) \supseteq \operatorname{null}(\psi) \neq 0$, which contradicts the assumption. Hence, they are linearly independent.

Solution 2: Under the isomorphism $\mathbb{F}^2 \cong (\mathbb{F}^2)'$, let φ be represented by (a_0, a_1) so $\varphi(x) = a_0x_1 + a_1x_2$. Let ψ be represented by (b_0, b_1). Then, the assumption $\operatorname{null}(\varphi) \cap \operatorname{null}(\psi) = \{0\}$ shows

$$\begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

if and only if $x = 0$. Therefore, the matrix has 2 pivots and (a_0, a_1), (b_0, b_1) are a basis for $\mathbb{F}^2$, so φ, ψ are for $(\mathbb{F}^2)'$.

Recap: Invariant Spaces

- Let $T \in \mathcal{L}(V)$ (a linear operator on V). A subspace $U \subseteq V$ is called T-invariant if $T(U) \subseteq U$.

- A 1D invariant space $\operatorname{span}\{v\}$ is special b/c $Tv = \lambda v$. If $\lambda \neq 0$, we call v an eigenvector & λ an eigenvalue.

- $E(\lambda, T) = \text{Null}(T - \lambda I)$ is called the eigenspace corresponding to λ, T.

Ex) $T = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ has $T - \lambda I = \begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix}$,

which is non-injective iff. $\lambda = 1, 2,$ or 3.

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e.g. $Te_1 = 3 \cdot e_1$.

- Our main calc. before introducing minimal polynomials is that eigenvectors associated to different eigenvalues are linearly independent.

Let $\{v_i\}$ be eigenvectors w.r.t. $\{\lambda_i\}$. Then, ~~are dependent~~

$$a_1 v_1 + \cdots + a_n v_n = 0$$

$$\Rightarrow a_1 T v_1 + \cdots + a_n T v_n = 0$$

$$\Rightarrow a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n = 0.$$

Thus, $a_2(\lambda_1 - \lambda_2)v_2 + \cdots + a_n(\lambda_1 - \lambda_n)v_n = 0$. If $\lambda_i \neq \lambda_j$ for $i \neq j$, we repeat this process to see $a_n(\lambda_1 - \lambda_n)\cdots(\lambda_{n-1} - \lambda_n)v_n = 0$, so $a_n = 0$.

Then,

$$\alpha_{n-1}(\lambda_1 - \lambda_{n-1}) \cdots (\lambda_{n-2} - \lambda_{n-1}) v_{n-1} + 0 = 0$$

gives $\alpha_{n-1} = 0$. Doing this n times shows $a_i = 0$ for all i